

**MECHANICAL ENGINEERING DEPARTMENT  
UNITED STATES NAVAL ACADEMY**

**EM423 - INTRODUCTION TO MECHANICAL VIBRATIONS**

**CONTINUOUS SYSTEMS - TORSIONAL VIBRATION IN SHAFTS**

**SYMBOLS**

$r$	Mass density
$I_P$	Polar moment of inertia
$G$	Shear modulus of elasticity
$q$	Angle of twist
$x$	Distance along the shaft
$T$	Internal torque (varies along the shaft)

**INTRODUCTION**

This theory is applicable to torsional vibrations in slender shafts. Some of the main areas where torsional vibrations can be a problem are long propulsion shafts, where under certain conditions the vibration can cause the engine control system to hunt. Torsional vibration in crankshafts in internal combustion engines can cause early fatigue failure. Torsion fatigue is one of the most common modes of mechanical failure.

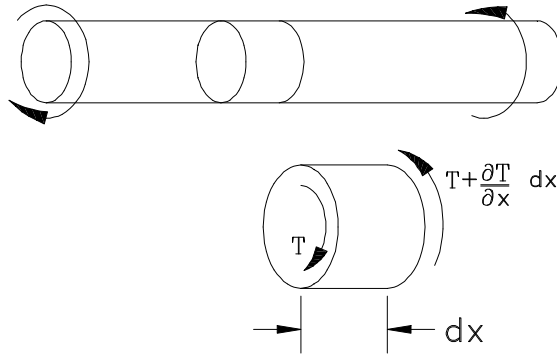
The mathematical derivation and resulting equation of motion are similar to those derived for longitudinal vibrations, but with a changed wave speed.

**ASSUMPTIONS**

1. The shaft is thin compared to its length.
2. The shaft is uniform, homogeneous and isotropic.
3. The material is within the elastic limit, and obeys Hooke's Law.
4. Plane sections remain plane.

**THEORY**

Consider a small element of shaft:



The net torque on the section is:

$$\left( T + \frac{\partial T}{\partial x} dx \right) - T = \frac{\partial T}{\partial x} dx$$

The angle of twist and internal torque are related by:

$$T = I_p G \frac{\partial q}{\partial x}$$

Differentiating and comparing to the previous result:

$$\text{net torque} = \frac{\partial T}{\partial x} dx = I_p G \frac{\partial^2 q}{\partial x^2} dx$$

Take moments about the axis of the shaft and equate the net torque to (mass moment of inertia)  $\times$  (angular acceleration)

$$I_p G \frac{\partial^2 q}{\partial x^2} dx = r I_p dx \frac{\partial^2 q}{dt^2}$$

$$\boxed{\begin{aligned} \frac{\partial^2 q}{\partial x^2} &= \frac{1}{c^2} \frac{\partial^2 q}{\partial t^2} \\ \text{with } c &= \sqrt{\frac{G}{r}} = \text{wave velocity} \end{aligned}}$$

This is the same wave equation as derived for string and rod vibrations, but with a different wave velocity. A general solution to this equation may therefore be written straight down as the product of a spatial and a time function:

$$q(x, t) = \left\{ A \sin\left(\frac{wx}{c}\right) + B \cos\left(\frac{wx}{c}\right) \right\} \sin(wt)$$

**NATURAL FREQUENCIES OF A FIXED-FREE SHAFT, length = L**

The general solution is:

$$q(x, t) = \left\{ A \sin\left(\frac{wx}{c}\right) + B \cos\left(\frac{wx}{c}\right) \right\} \sin(wt)$$

Apply the boundary conditions

At  $x = 0$ ;  $q = 0$  hence  $B = \text{zero}$

At  $x = L$ ;  $\frac{\partial q}{\partial x} = 0$

$$\frac{\partial q}{\partial x} = \frac{wA}{c} \cos\left(\frac{wx}{c}\right)$$

$$\text{so } \cos\left(\frac{wL}{c}\right) = 0$$

$$\text{and } \frac{w_n L}{c} = \left(n - \frac{1}{2}\right) \pi; \quad n = 1, 2, \dots$$

Rearrange for the natural frequencies:

$$w_n = \left(n - \frac{1}{2}\right) \frac{\pi c}{L} = \left(n - \frac{1}{2}\right) \frac{\pi}{L} \sqrt{\frac{G}{r}}$$

$$f_n = \frac{w_n}{2\pi} = \left(n - \frac{1}{2}\right) \frac{1}{2L} \sqrt{\frac{G}{r}}$$

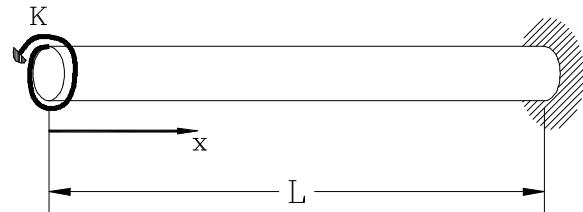
**DISCUSSION**

1. The solution for torsional vibrations in a shaft is similar to longitudinal vibrations of a rod, but with a different wave speed.
2. This means the same discussion applies to shaft vibrations. The only difference is that, for graphical presentation, angular displacements (rotations) are usually drawn at  $90^\circ$  to the axis of the shaft.

### ASSIGNMENTS

1. What is the difference between a string, a rod and a shaft?
2. Derive, from first principles, the speed of propagation of torsional strain waves along a uniform circular shaft.
3. Calculate the numerical value for the speed of propagation of torsional strain waves along a uniform circular steel shaft in m/s and miles/hr.
4. Determine an expression for the natural frequencies of torsional oscillations of a uniform shaft, length =  $L$ , clamped at the middle and free at the ends.

5. (extra credit) A uniform shaft has length  $L$ , density  $\rho$ , and torsional stiffness  $I_P G$ , where  $I_P$  is the polar moment of inertia of the cross section, and  $G$  is the shear modulus. The end  $x = 0$  is fastened to a torsional spring of stiffness  $K$  N-m/rad, and the end  $x = L$  is fixed. Determine the



frequency equation from which natural frequencies can be established. Verify your solution by finding the limits when  $K = 0$  and  $K = \infty$ , and comparing the equations to those for a fixed-free and fixed-fixed shaft respectively.

**SOLUTIONS**

1. What is the difference between a string, a rod and a shaft?

This is discussed in class. A rod is a long, thin structure, and we are considering longitudinal (compression) waves traveling along it. A shaft is a long, thin structure, and we are considering torsional waves traveling along it. A string is a long, thin structure that has very little flexural rigidity - you can't push a piece of string. Depending on the situation and the particular vibration, a long, thin piece of metal could be considered as a shaft, rod or string, or even all three at the same time. We are unlikely to get torsional or compression waves in a string

2. Derive, from first principles, the speed of propagation of torsional strain waves along a uniform circular shaft.

See the course handout

*CONTINUOUS SYSTEMS - TORSIONAL VIBRATION IN SHAFTS*

3. Calculate the numerical value for the speed of propagation of torsional strain waves along a uniform circular steel shaft in m/s and miles/hr.

$$c = \sqrt{\frac{G}{r}} = \sqrt{\frac{81 \times 10^9}{7843}} = 3,214 \text{ m/s}$$

$$= 3214 \left[ \frac{m}{s} \right] \left[ \frac{1 \text{ in}}{0.0254 \text{ m}} \right] \left[ \frac{1 \text{ ft}}{12 \text{ in}} \right] \left[ \frac{1 \text{ mile}}{5280 \text{ ft}} \right] \left[ \frac{3600 \text{ s}}{1 \text{ hour}} \right] = 7,190 \text{ miles/hour}$$

4. Determine an expression for the natural frequencies of torsional oscillations of a uniform shaft, length =  $L$ , clamped at the middle and free at the ends.

$$q(x, t) = \left\{ A \sin\left(\frac{wx}{c}\right) + B \cos\left(\frac{wx}{c}\right) \right\} \sin(wt)$$

$$\frac{\partial q}{\partial x} = \left\{ A \cos\left(\frac{wx}{c}\right) - B \sin\left(\frac{wx}{c}\right) \right\} \left(\frac{w}{c}\right) \sin(wt)$$

At  $x = 0$  (middle of shaft);  $q = 0$  hence  $B = \text{zero}$

$$\text{At } x = \frac{L}{2}; \quad \frac{\partial q}{\partial x} = 0$$

This is only true for at time if  $\cos\left(\frac{wL}{2c}\right) = 0$

$$\text{so } \left(\frac{w_n L}{2c}\right) = \left(n - \frac{1}{2}\right) \pi \quad n = 1, 2, 3, \dots$$

$$\text{from which } w_n = 2\pi f_n = \frac{(2n-1)\pi c}{L} = \frac{(2n-1)\pi}{L} \sqrt{\frac{G}{r}}$$

$$\text{and } f_n = \frac{(2n-1)}{2L} \sqrt{\frac{G}{r}}$$

5. (extra credit) A uniform shaft has length  $L$ , density  $r$ , and torsional stiffness  $I_P G$ , where  $I_P$  is the polar moment of inertia of the cross section, and  $G$  is the shear modulus. The end  $x = 0$  is fastened to a torsional spring of stiffness  $K$  N-m/rad, and the end  $x = L$  is fixed. Determine the frequency equation from which natural frequencies can be established. Verify your solution by finding the limits when  $K = 0$  and  $K = \infty$ , and comparing the equations to those for a fixed-free and fixed-fixed shaft respectively.

*The solution to extra credit problems is available from your instructor.*